

MARTIN BOUNDARY OF A FINE DOMAIN AND A FATOU-NAÏM-DOOB THEOREM FOR FINELY SUPERHARMONIC FUNCTIONS

MOHAMED EL KADIRI AND BENT FUGLEDE

ABSTRACT. We construct the Martin compactification \overline{U} of a fine domain U in \mathbb{R}^n ($n \geq 2$) and the Riesz-Martin kernel K on $U \times \overline{U}$. We obtain the integral representation of finely superharmonic functions ≥ 0 on U in terms of K and establish the Fatou-Naim-Doob theorem in this setting.

1. INTRODUCTION

The fine topology on an open set $\Omega \subset \mathbb{R}^n$ was introduced by H. Cartan in classical potential theory. It is defined as the smallest topology on Ω making every superharmonic function on Ω continuous. This topology is neither locally compact nor metrizable. The fine topology has, however, other good properties which allowed the development in the 1970's of a 'fine' potential theory on a finely open set $U \subset \Omega$, starting with the book [16] of the second named author. The harmonic and superharmonic functions and the potentials in this theory are termed finely [super]harmonic functions and fine potentials. Generally one distinguishes by the prefix 'fine(ly)' notions in fine potential theory from those in classical potential theory on a usual (Euclidean) open set. Very many results from classical potential theory have been extended to fine potential theory. In this article we study the invariant functions, generalizing the non-negative harmonic functions in the classical Riesz decomposition theorem; and the integral representation of finely superharmonic functions in terms of the 'fine' Riesz-Martin kernel. The Choquet representation theorem plays a key role. We close by establishing the Fatou-Naim-Doob theorem on the fine limit of finely superharmonic functions at the fine Martin boundary, inspired in particular by the axiomatic approach of Taylor [31].

In forthcoming continuations [14] and [15] of the present paper we study sweeping on a subset of the Riesz-Martin space, and the Dirichlet problem at the Martin boundary of U .

Speaking in slightly more detail we consider the standard H -cone $\mathcal{S}(U)$ of all finely superharmonic functions ≥ 0 on a given fine domain U (that is, a finely connected finely open subset of a Greenian domain $\Omega \subset \mathbb{R}^n$). Generalizing the classical Riesz representation theorem it was shown in [21], [22] that every

function $u \in \mathcal{S}(U)$ has a unique integral representation of the form

$$u(x) = \int G_U(x, y) d\mu(y) + h(x), \quad x \in \Omega,$$

where μ is a (positive) Borel measure on U , G_U is the (fine) Green kernel for U , and h is an invariant function on U . The term ‘invariant’ reflects a property established in [21, Theorem 4.4] and generalizing fine harmonicity.

An interesting problem is whether every minimal invariant function is finite valued and hence finely harmonic on U , or equivalently whether every invariant function is the sum of a sequence of finely harmonic functions. A negative answer to this question was recently obtained (in dimension $n > 2$) by Gardiner and Hansen [23].

Recall from [16, Theorem 8.1] that a function $u : U \mapsto]-\infty, +\infty]$ is said to be *finely hyperharmonic* if (i) u is finely l.s.c. and (ii) the induced fine topology on U has a base consisting of finely open sets V of fine closure $\tilde{V} \subset U$ such that

$$(1.1) \quad u(x) \geq \int^* u d\varepsilon_x^{\mathbb{C}V}$$

for every $x \in V$ (complements being taken relative to Ω). Recall that the swept measure $\varepsilon_x^{\mathbb{C}V}$ (serving as a fine harmonic measure) is carried by the fine boundary $\partial_f V \subset U$ and does not charge the polar sets. As shown by Lyons [29], condition (ii) can be replaced equivalently by the requirement that every point $x \in U$ has a fine neighborhood base consisting of finely open sets V with $\tilde{V} \subset U$ such that (1.1) holds. Every finely hyperharmonic function is *finely continuous* ([16, Theorem 9.10]).

A finely hyperharmonic function u on U is said to be *finely superharmonic* if u is not identically $+\infty$ on any fine component of U , or equivalently if u is finite on a finely dense subset of U , or still equivalently if $u < +\infty$ quasi-everywhere (q.e.) on U . We denote by $\mathcal{S}(U)$ the convex cone of all *positive* finely superharmonic functions on U , which is henceforth supposed to be finely connected (a fine domain).

In [12] the first named author has defined a topology on the cone $\mathcal{S}(U)$, generalizing the topology of R.-M. Hervé in classical potential theory. By identifying this topology with the natural topology, now on the standard H -cone $\mathcal{S}(U)$, it was shown in [12] that $\mathcal{S}(U)$ has a compact base B , and by Choquet’s theorem that every function $u \in \mathcal{S}(U)$ admits a unique integral representation of the form

$$u(x) = \int_B s(x) d\mu(s), \quad x \in U,$$

where μ is a finite measure on B carried by the (Borel) set of all extreme finely superharmonic functions belonging to B .

In the present article we define the Martin compactification \overline{U} and the Martin boundary $\Delta(U)$ of U . While the Martin boundary of a usual open set is closed and hence compact, all we can say in the present setup is that $\Delta(U)$ is a G_δ subset of the compact Riesz-Martin space $\overline{U} = U \cup \Delta U$ endowed with the natural topology. Nevertheless we can define a Riesz-Martin kernel $K : U \times \overline{U} \rightarrow]0, +\infty]$ with good properties of lower semicontinuity and measurability. Every function $u \in \mathcal{S}(U)$ has an integral representation $u(x) = \int_{\overline{U}} K(x, Y) d\mu(Y)$ in terms of a Radon measure μ on \overline{U} . This representation is unique if it is required that μ be carried by $U \cup \Delta_1(U)$ where $\Delta_1(U)$ denotes the minimal Martin boundary of U , which is likewise a G_δ in \overline{U} . In that case we write $\mu = \mu_u$. It is shown that, for any Radon measure μ on \overline{U} , the associated function $u = \int K(\cdot, Y) d\mu(Y) \in \mathcal{S}(U)$ is a fine potential, resp. an invariant function, if and only if μ is carried by U , resp. $\Delta(U)$.

There is a notion of minimal thinness of a set $E \subset U$ at a point $Y \in \Delta_1(U)$, and an associated minimal-fine filter $\mathcal{F}(Y)$. As a generalization of the classical Fatou-Naïm-Doob theorem we show that for any finely superharmonic function $u \geq 0$ on U and for μ_1 -almost every point $Y \in \Delta_1(U)$, $u(x)$ has the limit $(d\mu_u/d\mu_1)(Y)$ as $x \rightarrow Y$ along the minimal-fine filter $\mathcal{F}(Y)$. Here $d\mu_u/d\mu_1$ denotes the Radon-Nikodým derivative of the absolutely continuous component of μ_u with respect to the absolutely continuous component of the measure μ_1 representing the constant function 1, which is finely harmonic and hence invariant. Actually, we establish for any given invariant function $h > 0$ the more general h -relative version of this result.

Notations: For a Greenian domain $\Omega \subset \mathbb{R}^n$ we denote by G_Ω the Green kernel for Ω . If U is a fine domain in Ω we denote by $\mathcal{S}(U)$ the convex cone of finely superharmonic functions ≥ 0 on U in the sense of [16]. The convex cone of fine potentials on U (that is, the functions in $\mathcal{S}(U)$ for which every finely subharmonic minorant is ≤ 0) is denoted by $\mathcal{P}(U)$. The cone of invariant functions on U is denoted by $\mathcal{H}_i(U)$; it is the orthogonal band to $\mathcal{P}(U)$ relative to $\mathcal{S}(U)$. By G_U we denote the (fine) Green kernel for U , cf. [18]. If $A \subset U$ and $f : A \rightarrow [0, +\infty]$ one denotes by R_f^A , resp. \widehat{R}_f^A , the reduced function, resp. the swept function, of f on A relative to U , cf. [16, Section 11]. For any set $A \subset \Omega$ we denote by \widetilde{A} the fine closure of A in Ω , and by $b(A)$ the base of A in Ω , that is, the G_δ set of points of Ω at which A is not thin, in other words the set of all fine limit points of A in Ω . We define $r(A) = \mathcal{C}b(\mathcal{C}A)$ (complements and bases relative to Ω). Thus $r(A)$ is a K_σ set, the least regular finely open subset of Ω containing the fine interior A' of A , and $r(A) \setminus A$ is polar.

Acknowledgment. The Authors thank a referee for valuable references to work related to Theorem 2.8 and Corollary 2.9, as described in Remark 2.10.

2. THE NATURAL TOPOLOGY ON THE CONE $\mathcal{S}(U)$

We begin by establishing some basic properties of invariant functions on an arbitrary fine domain $U \subset \Omega$ (Ω a Greenian domain in \mathbb{R}^n , $n \geq 2$). First an auxiliary lemma of a general nature:

Lemma 2.1. *Every finely continuous function $f : U \rightarrow \overline{\mathbb{R}}$ is Borel measurable in the relative Euclidean topology on $U \subset \Omega$.*

Proof. We shall reduce this to the known particular case where $U = \Omega$, see [17]. Since $\overline{\mathbb{R}}$ is homeomorphic with $[0, 1]$ we may assume that $f(U) \subset [0, 1]$. Recall that the base $b(\varphi)$ of a function $\varphi : \Omega \rightarrow [0, 1]$ is defined as the finely derived function $b(\varphi) : \Omega \rightarrow [0, 1]$:

$$b(\varphi)(x) = \text{fine } \limsup_{y \rightarrow x, y \in \Omega \setminus x} \varphi(y)$$

for $x \in \Omega$, see [10, p. 590], [17]. Extend the given finely continuous function $f : U \rightarrow [0, 1]$ to functions $f_0, f_1 : \Omega \rightarrow [0, 1]$ by defining $f_0 = 0$, $f_1 = 1$ on $\Omega \setminus U$. Then f_1 and $1 - f_0$ are finely u.s.c. on Ω , and hence $b(f_1) \leq f_1$ and $b(1 - f_0) \leq 1 - f_0$. Since $f_0 \leq f_1$ it follows that $f_0 \leq b(f_0) \leq b(f_1) \leq f_1$. But $f_0 = f_1 = f$ on U , and so $f = b(f_0) = b(f_1)$ on U . According to [17], $b(f_1)$ (and $b(1 - f_0)$) is of class $\mathcal{G}_\delta(\Omega)$, that is, representable as the pointwise infimum of a (decreasing) sequence of Euclidean l.s.c. functions $\Omega \rightarrow [0, +\infty]$. Consequently, f (and $1 - f$) is of class $\mathcal{G}(U)$, that is representable as the pointwise infimum of a sequence of l.s.c. functions $U \rightarrow [0, +\infty]$, U being given the relative Euclidean topology. In particular, f is Borel measurable in that topology on U . \square

Lemma 2.2. *If h is invariant on U , if $u \in \mathcal{S}(U)$, and if $h \leq u$, then $h \preceq u$.*

Proof. There is a polar set $E \subset U$ such that h is finely harmonic on $U \setminus E$, and hence $u - h$ is finely superharmonic on $U \setminus E$. According to [16, Theorem 9.14], $u - h$ extends by fine continuity to a function $s \in \mathcal{S}(U)$ such that $h + s = u$ on $U \setminus E$ and hence on all of U , whence $h \preceq u$. \square

Lemma 2.3. *If $u \in \mathcal{S}(U)$ and $A \subset U$ then $R_u^A(x) = \int_U u d\varepsilon_x^{A \cup (\Omega \setminus U)}$ for $x \in U$.*

Proof. The integral exists because $u \geq 0$ is Borel measurable in the relative Euclidean topology on $U \subset \Omega$ by Lemma 2.1. Since \widehat{R}_u^A and $\int_U u d\varepsilon_x^{A \cup (\Omega \setminus U)}$ remain unchanged if A is replaced by its base $b(A) \cap U$ relative to U and since the base operation is idempotent we may assume that $A = b(A) \cap U$. Consequently, $\widehat{R}_u^A = R_u^A$ and the set $V := U \setminus A$ is finely open (and regular). Let u_0 denote the extension of u to $\widetilde{U} = U \cup \partial_f U$ by the value 0 on $\partial_f U$. Every

function $s \in \mathcal{S}(U)$ with $s \geq u$ on A is an upper function (superfunction) for u_0 relative to V , see [16, §§14.3–14.6] concerning the (generalized) fine Dirichlet problem. It follows that

$$\widehat{R}_u^A(x) = R_u^A(x) \geq \overline{H}_{u_0}^V(x) = \int_{\Omega}^* u_0 d\varepsilon_x^{\Omega \setminus V} = \int_U^* u d\varepsilon_x^{A \cup (\Omega \setminus U)}$$

for $x \in V$. For the opposite inequality, consider any upper function v for u_0 relative to V . In particular, $v \geq -p$ on V for some finite and hence semi-bounded potential p on Ω . Define $w = u \wedge v$ on V and $w = u$ on $A = U \setminus V$. Then

$$\liminf_{y \rightarrow x, y \in V} w(y) \geq u(x) \wedge \liminf_{y \rightarrow x, y \in V} v(y) = u(x)$$

for every $x \in U \cap \partial_f V$. It therefore follows by [16, Lemma 10.1] that w is finely hyperharmonic on U . Moreover, w is an upper function for u relative to V because $w = u \wedge v \geq 0 \wedge (-p) = -p$ on V . Since $w = u$ on A we have $w \geq \widehat{R}_u^A$ on U , and in particular $v \geq w \geq \widehat{R}_u^A$ on V . By varying v we obtain $\overline{H}_{u_0}^V \geq \widehat{R}_u^A \geq R_u^A$ on V . Altogether we have established the asserted equality for $x \in V$. It also holds for $x \in U \setminus V = A = b(A) \cap U$ because $\widehat{R}_u^A(x) = R_u^A(x) = u(x)$ and because $\varepsilon_x^{A \cup (\Omega \setminus U)} = \varepsilon_x$, noting that $x \in b(A) \subset b(A \cup (\Omega \setminus U))$. \square

Lemma 2.4. *Let $u \in \mathcal{S}(U)$ and let A be a subset of U . The restriction of \widehat{R}_u^A to any finely open subset V of $U \setminus A$ is invariant.*

Proof. We have $\widehat{R}_u^A = \sup_{k \in \mathbb{N}} \widehat{R}_{u \wedge k}^A$. Each of the functions $\widehat{R}_{u \wedge k}^A$ is finely harmonic on V by [16, Corollary 11.13], and hence \widehat{R}_u^A is invariant on V in view of Lemma 2.2 because the invariant functions on V form a band (the orthogonal band to $\mathcal{P}(V)$). \square

As shown by the following example, the invariant functions on finely open subsets of U do not form a sheaf.

Example 2.5. Let μ be the one-dimensional Lebesgue measure on a line segment E in $\Omega := U := \mathbb{R}^3$. Then E is polar and hence everywhere thin in \mathbb{R}^3 . Every point $x \in \mathbb{R}^3$ therefore has a fine neighborhood V_x with $\mu(V_x) = 0$. (For $x \in E$ take $V_x = \{x\} \cup \mathbb{C}L$, where L denotes the whole line extending E .) Thus μ does not have a (minimal) fine support (unlike measures which do not charge any polar set). The Green potential $G_{\Omega}\mu$ is invariant on each V_x (but of course not on Ω). For if p denotes a non-zero fine potential on V_x , $x \in E$, such that $p \preccurlyeq G_{\Omega}\mu$ on V_x then p is finely harmonic on $V_x \setminus \{x\} = \mathbb{C}L$ along with $G_{\Omega}\mu$. Hence p behaves on V_x near x like a constant times $G_{V_x}\varepsilon_x$ and is therefore of the order of magnitude $1/r$, where r denotes the distance from x , cf. [18, Théorème]. But on the plane through x orthogonal to L , $G_{\Omega}\mu$ behaves like a constant times $\log(1/r)$, in contradiction with $p \leq G_{\Omega}\mu$.

The invariant functions on finely open subsets of U do, however, have a kind of countable sheaf property, as shown by (a) and (b) in the following theorem:

Theorem 2.6. (a) *Let $u \in \mathcal{S}(U)$ be invariant and let V be a finely open subset of U . Then $u|_V$ is invariant.*

(b) *Let $u \in \mathcal{S}(U)$, and let (U_j) be a countable cover of U by regular finely open subsets of U . If each $u|_{U_j}$ is invariant then u is invariant.*

(c) *Let (u_α) be a decreasing net of invariant functions in $\mathcal{S}(U)$. Then $\widehat{\inf_\alpha u_\alpha}$ is invariant. Moreover, the set $V = \{\inf_\alpha u_\alpha < +\infty\}$ is finely open, and $\widehat{\inf_\alpha u_\alpha} = \inf_\alpha u_\alpha$ on V .*

Proof. (a) Let $p \in \mathcal{P}(V)$ satisfy $p \preceq u$ on V . Write $p = G_V \mu$ (considered on V) in terms of the associated Borel measure μ on Ω such that the inner measure $\mu_*(\mathbb{C}V) = 0$. We shall prove that $p = 0$. An extension of μ from V to a larger subspace of Ω by 0 off V will also be denoted by μ . Thus $p = G_V \mu$ extends to the fine potential $G_{U \cap r(V)} \mu$ on $U \cap r(V)$ and further extends to a finely continuous function $f : U \rightarrow [0, +\infty]$ which equals 0 off $U \cap r(V)$.

Suppose to begin with that μ is finite and (after extension to Ω) carried by a Euclidean compact subset K of Ω contained in $r(V)$. Let $q := \widehat{R}_f$ (sweeping relative to U). Then $f \leq G_U \mu$ and hence $q \leq G_U \mu < +\infty$ q.e. on U , and so q is a fine potential on U along with $G_U \mu$. On the other hand, $q \leq u$ because $f \leq u$ on U . In the first place, $f = p \leq u$ on V and hence $f \leq u$ on $U \cap r(V) \subset U \cap \widetilde{V}$ by fine continuity of f and u . Secondly, $f = 0$ on $U \setminus r(V)$. Since $p \preceq u$ on V we have $u = p + s$ on V for a certain $s \in \mathcal{S}(V)$. Since $U \cap r(V) \setminus V$ is polar, s extends by fine continuity to a similarly denoted $s \in \mathcal{S}(U \cap r(V))$, and we have $u = G_{U \cap r(V)} \mu + s$ on $U \cap r(V)$. Since $\text{fine lim } G_{U \cap r(V)} \mu = f = 0$ at $U \cap \partial_f r(V) \subset U \setminus r(V)$ we have $\text{fine lim } s = u$ at $U \cap \partial_f r(V)$. It follows by [16, Lemma 10.1] that the extension of s to U by u on $U \setminus r(V)$ is of class $\mathcal{S}(U)$. Denoting also this extension by s we have $u = s + f$ on U . By Mokobodzki's inequality in our setting, see [16, Lemma 11.14], we infer that $q = \widehat{R}_f \preceq u$. Since u is invariant on U and $q \in \mathcal{P}(U)$ it follows that $q = 0$ and hence $p \leq q = 0$ on V , showing that indeed $u|_V$ is invariant.

Dropping the above temporary hypothesis that μ be finite and carried by a Euclidean compact subset of $r(V)$ we decompose μ in accordance with [21, Lemma 2.3] into the sum of a sequence of finite measures μ_j with Euclidean compact supports $K_j \subset r(V)$. Since $G_V \mu_j \preceq G_V \mu = p \preceq u|_V$, the result of the above paragraph applies with μ replaced by μ_j . It follows that $G_V \mu_j = 0$ and hence $p = G_V \mu = \sum_j G_V \mu_j = 0$. Thus \widehat{R}_u^A is indeed invariant on V .

(b) For each index j there is by [21, Theorem 4.4] a countable finely open cover $(V_{jk})_k$ of U_j such that $\widetilde{V}_{jk} \subset r(U_j) = U_j$ and (with sweeping relative to

$U_j)$

$$\widehat{R}_{u|U_j}^{U_j \setminus V_{jk}} = u|_{U_j}$$

for each k . It follows that (with sweeping relative to U , resp. U_j)

$$u \geq \widehat{R}_u^{U \setminus V_{jk}} \geq \widehat{R}_{u|U_j}^{U_j \setminus V_{jk}} = u$$

on each U_j , so equality prevails here. In particular, $u = \widehat{R}_u^{U \setminus V_{jk}}$ on V_{jk} for each j, k . Consequently, u is invariant according to the quoted theorem applied to the countable cover $(V_{jk})_{jk}$ of U .

(c) For indices α, β with $\alpha < \beta$ we have $u_\beta \leq u_\alpha$ and hence $u_\beta \preccurlyeq u_\alpha$ by Lemma 2.2. The claim therefore follows from [16, c), p. 132]. \square

Throughout the rest of the article, U is supposed (in the absence of other indication) to be a *regular* fine domain in the Greenian domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. In particular, U is a Euclidean K_σ subset of Ω . We proceed to introduce and study the natural topology on the H -cone $\mathcal{S}(U)$ of non-negative finely superharmonic functions on U .

Theorem 2.7. *There exists a resolvent family (W_λ) of kernels on U which are absolutely continuous with respect to a measure σ on U such that $\mathcal{S}(U)$ is the cone of excessive functions which are finite σ -a.e.*

Proof. Let $p = G_\Omega \tau$ be a strict bounded continuous potential on the Greenian domain Ω in \mathbb{R}^n . Then the measure τ does not charge the polar sets and we have $\tau(\omega) > 0$ for any fine open subset of Ω . Denote by V the Borel measurable kernel on Ω defined by

$$Vf(x) = \int G_\Omega(x, y)f(y)d\tau(y)$$

for any Borel measurable function $f \geq 0$ in Ω and for $x \in \Omega$, and by (V_λ) the resolvent family of kernels whose kernel potential is V . According to [8, Proposition 10.2.2, p. 248], the cone of excessive functions of the resolvent (V_λ) is the cone $\mathcal{S}(\Omega) \cup \{+\infty\}$ (and hence $\mathcal{S}(\Omega)$ is the cone of excessive functions of (V_λ) which are finite τ -a.s). Define a kernel W on U by

$$Wf = V\bar{f} - \widehat{R}_{V\bar{f}}^{\mathcal{C}U}$$

(restricted to U) for any Borel measurable function $f \geq 0$ in U , where \bar{f} denotes the extension of f to Ω by 0 in $\Omega \setminus U$. Then by [7, Theorem 2.5] there exists a unique resolvent family (W_λ) of Borel measurable kernels having the potential kernel W .

We proceed to determine the excessive functions for the resolvent (W_λ) . Every superharmonic function $s \geq 0$ on $\mathcal{S}(\Omega)$ is excessive for the resolvent V_λ , and hence there exists by [9, Théorème 17, p. 11] an increasing sequence (f_j) of bounded Borel measurable functions ≥ 0 such that $s = \sup V f_j$. It follows

that $s - \widehat{R}_s^{\mathcal{C}U} = \sup_j W(g_j)$, where g_j denotes the restriction of f_j to U . This shows that $s - \widehat{R}_s^{\mathcal{C}U}$ is excessive for (W_λ) . For any $u \in \mathcal{S}(U) \cup \{+\infty\}$ there exists by [20, Theorem 3] a sequence of superharmonic functions $s_j \geq 0$ on Ω such that

$$u = \sup_j (s_j - \widehat{R}_{s_j}^{\mathcal{C}U}),$$

where the sequence $(s_j - \widehat{R}_{s_j}^{\mathcal{C}U})$ is increasing and hence u is excessive for (W_λ) . Conversely, let u be excessive for (W_λ) . According to [9, Théorème 17, p. 11] there exists an increasing sequence of potentials (Wf_j) of bounded Borel measurable functions ≥ 0 such that $u = \sup_j Wf_j$. For each j we have $W(f_j) = V(\bar{f}_j) - \widehat{R}_{V(\bar{f}_j)}^{\mathcal{C}U}$. But $V(\bar{f}_j)$ is finite and continuous on U , and so $\widehat{R}_{V(\bar{f}_j)}^{\mathcal{C}U}$ is finely harmonic on U . It follows that $W(f_j) \in \mathcal{S}(U)$. Consequently, u is finely hyperharmonic on U , that is, $u \in \mathcal{S}(U) \cup \{+\infty\}$.

Let σ be the restriction of the measure τ to U . Then for any $A \in \mathcal{B}(U)$ (the finely Borel σ -algebra on U) such that $\sigma(A) = 0$ we have $W1_A = (V\overline{1_A})|_U = 0$, hence the resolvent (W_λ) is absolutely continuous with respect to σ . Since τ does not charge the polar sets, we see that $\mathcal{S}(U)$ is the cone of excessive functions which are finite σ -a.e. This completes the proof of Theorem 2.7. \square

It follows from Theorem 2.7 by [6, Theorem 4.4.6] that $\mathcal{S}(U)$ is a standard H -cone of functions on U . Following [6, Section 4.3] we give $\mathcal{S}(U)$ the natural topology. This topology on $\mathcal{S}(U)$ is metrizable and induced by the weak topology on a locally convex topological vector space in which $\mathcal{S}(U)$ is embedded as a proper convex cone. This cone is well-capped with compact caps, but we show that the cone $\mathcal{S}(U)$ even has a compact base, and that is crucial for our investigation. We shall need the following results from [12]:

Theorem 2.8. [12, Lemme 3.5]. *There exists a sequence (K_j) of Euclidean compact subsets of Ω contained in U and a polar set $P \subset U$ such that*

1. $U = P \cup \bigcup_j K'_j$, where K'_j denotes the fine interior of K_j .
2. For any j the restriction of any function from $\mathcal{S}(U)$ to K_j is l.s.c. in the Euclidean topology.

Corollary 2.9. *There exists a sequence (H_j) of Euclidean compact subsets of U , each non-thin at any of its points, and a polar set P such that*

1. $U = P \cup \bigcup_j H_j$.
2. For any j the restriction of any function from $\mathcal{S}(U)$ to H_j is l.s.c. in the Euclidean topology.

Proof. Write $U = P \cup \bigcup_j K'_j$ as in Theorem 2.8. Recall that the fine interior of a subset of Ω is regular. For each j let (U_j^m) denote the fine components of K'_j ; they are likewise regular. For each couple (j, m) let $y_{j,m}$ be a point of U_j^m . And for each integer $n > 0$ put $H_{j,m,n} = \{x \in U_j^m : G_{U_j^m}(x, y_{j,m}) \geq \frac{1}{n}\}$. The

sets $H_{j,m,n}$ are Euclidean compact and non-thin at any of its points (in view of [16, Theorem 12.6]), and we have $U_j^m = \bigcup_n H_{j,m,n}$. The sequence $(H_{j,m,n})$ and the polar set P have the stated properties. \square

Remark 2.10. The existence of a sequence (K_j) of compact subsets of U and a polar set P with $U = P \cup \bigcup_j K_j$ such that a given finely continuous function (in particular a superharmonic function) is continuous relative to each K_j follows from the pioneering work of Le Jan [26, 27, 28], which applies more generally to the excessive functions of the resolvent associated with the Hunt process. The weaker form of 1. in our Theorem 2.8 in which $U = P \cup \bigcup_j K'_j$ is replaced by the condition $U = P \cup \bigcup_j K_j$, is a consequence of [5, Corollaire 1.6] together with the existence of a family of universally continuous elements which is increasingly dense in $\mathcal{S}(U)$. In the present case our Theorem 2.8 is stronger than that of Beznea and Boboc. In fact, our result is not a consequence of that of Beznea and Boboc because for a nest (K_j) of U the set $\bigcup_j K_j \setminus \bigcup_j K'_j$ is not necessarily polar, as it is seen by the following example:

Example. Let A be a compact non-polar subset of Ω with empty fine interior (for example A can be a compact ball in some hyperplane in \mathbb{R}^n such that $A \subset \Omega$). Let $\Omega_1 = \Omega \setminus A$. Then Ω_1 is open and there exists an increasing sequence (B_j) of open subsets of Ω_1 such that $\overline{B_j} \subset \Omega_1$ for every j ($\overline{B_j}$ denoting the Euclidean closure of B_j) and that $\bigcup_j B_j = \Omega_1$. For any j write $K_j = \overline{B_j} \cup A$. Clearly, (K_j) is an increasing sequence of compact subsets of Ω with $\bigcup_j K_j = \bigcup_j \overline{B_j} \cup A = \Omega_1 \cup A = \Omega$. It suffices to show that $K'_j \subset \overline{B_j}$ for every j , for then $\bigcup_j K'_j \subset \bigcup_j \overline{B_j} = \Omega_1 = \Omega \setminus A$ with A non-polar. Let $x \in K'_j$. If $x \in A$ then $V := \Omega \setminus \overline{B_j}$ is an open neighborhood of x and $V \cap \overline{B_j} = \emptyset$. On the other hand, $W := K'_j$ is a fine neighborhood of x contained in K_j . Then $W \cap V \subset K_j$ and $(W \cap V) \cap \overline{B_j} = \emptyset$, hence $x \in W \cap V \subset A$. But $W \cap V$ is finely open and $A' = \emptyset$, so actually $x \notin A$, and since $x \in K'_j \subset K_j = \overline{B_j} \cup A$ we have $x \in \overline{B_j}$. Because this holds for every $x \in K'_j$ we indeed have $K'_j \subset \overline{B_j}$.

Remark 2.11. One may recover Corollary 2.9 from [5, Corollaire 1.6]. In fact, let (K_j) be a sequence of compact subsets of U and let A be a polar set with $U = A \cup \bigcup_j K_j$ such that the restriction of any function $u \in \mathcal{S}(U)$ to each K_j is l.s.c. One may suppose that all the compact sets K_j are non-polar. By repeated application of Ancona's theorem [3] it follows that each K_j is the union of a polar set A_j and sequence $(K_{j,k})_k$ of compact sets $K_{j,k}$, each of which is non-thin at each of its points. The double sequence $(K_{j,k})_{j,k}$, arranged as a single sequence (H_l) , together with the polar set $P := A \cup \bigcup_j A_j$, meet the requirements in Corollary 2.9.

We shall now use the sequence (H_j) from this corollary to define in analogy with [30] a locally compact topology on the cone $\mathcal{S}(U)$. For each j let $\mathcal{C}_l(H_j)$

denote the space of l.s.c. functions on H_j with values in $\overline{\mathbb{R}}_+$, and provide this space with the topology of convergence in graph (cf. [30]). It is known that $\mathcal{C}_l(H_j)$ is a compact metrizable space in this topology. Let d_j denote a distance compatible with this topology. We define a pseudo-distance d on $\mathcal{S}(U) \cup \{+\infty\}$ by

$$d(u, v) = \sum_j \frac{1}{2^j \delta(\mathcal{C}_l(H_j))} d_j(u|_{H_j}, v|_{H_j})$$

for each couple (u, v) of functions from $\mathcal{S}(U) \cup \{+\infty\}$, where $\delta(\mathcal{C}_l(H_j))$ denotes the diameter of $\mathcal{C}_l(H_j)$. Since two finely hyperharmonic functions are identical if they coincide quasi-everywhere it follows that d is a true distance on $\mathcal{S}(U) \cup \{+\infty\}$. We denote by \mathcal{T} the topology on $\mathcal{S}(U) \cup \{+\infty\}$ defined by the distance d .

For any filter \mathcal{F} on $\mathcal{S}(U) \cup \{+\infty\}$ we write

$$\liminf_{\mathcal{F}} = \sup_{M \in \mathcal{F}} \inf_{u \in M} \widehat{u},$$

where the l.s.c. regularized $\widehat{\inf}_{u \in M} u$ is taken with respect to the fine topology.

Theorem 2.12. [12, Théorème 3.6]. *The cone $\mathcal{S}(U) \cup \{+\infty\}$ is compact in the topology \mathcal{T} . For any convergent filter \mathcal{F} on $\mathcal{S}(U) \cup \{+\infty\}$ we have*

$$\lim_{\mathcal{F}} = \liminf_{\mathcal{F}}.$$

Proof. Let \mathcal{U} be an ultrafilter on $\mathcal{S}(U) \cup \{+\infty\}$. For any $M \in \mathcal{U}$ put $u_M = \inf_{u \in M} u$. For each j the ultrafilter base \mathcal{U}_j obtained from \mathcal{U} by taking restrictions to the Euclidean compact H_j from Corollary 2.9, converges in the compact space $\mathcal{C}_l(H_j)$ to the function $u_j := \sup_{M \in \mathcal{U}} \widehat{u_M}^j$ (where \widehat{v}^j for $v \in \mathcal{S}(U) \cup \{+\infty\}$ denotes the finely l.s.c. regularized of the restriction of v to H_j). The finely l.s.c. regularized $\widehat{u_M}$ of u_M in U is l.s.c. in H_j by Theorem 2.8 and minorizes u_M , whence $\widehat{u_M} \leq \widehat{u_M}^j$. On the other hand there exists a polar set $A \subset \Omega$ such that $u_M = \widehat{u_M}$ in $U \setminus A$, and so $\widehat{u_M}^j \leq \widehat{u_M}$ in $H_j \setminus A$. But for $x \in A$ we have $\widehat{u_M}^j(x) \leq \widehat{u_M}(x)$ because $\widehat{u_M}$ is finely continuous on U and x is in the fine closure of $H_j \setminus A$ since H_j is non-thin at x . We conclude that $u_j = \liminf_{\mathcal{U}} u$ in H_j for each j . Since the function $u := \liminf_{\mathcal{U}} u$ belongs to $\mathcal{S}(U) \cup \{+\infty\}$ according to [16, §12.9] it follows that the filter \mathcal{U} converges to u in the topology \mathcal{T} . This proves that $\mathcal{S}(U) \cup \{+\infty\}$ is compact in the topology \mathcal{T} . \square

Corollary 2.13. *The topology of convergence in graph coincides with the natural topology on $\mathcal{S}(U)$.*

Proof. This follows immediately from Theorem 2.12 and [6, Theorem 4.5.8]. \square

Corollary 2.14. *The cone $\mathcal{S}(U)$ endowed with the natural topology has a compact base.*

Proof. From Theorem 2.12 and Corollary 2.13 it follows that the natural topology on $\mathcal{S}(U)$ is locally compact, and we infer by a theorem of Klee [2, Theorem II.2.6] that indeed $\mathcal{S}(U)$ has a compact base. \square

Corollary 2.15. *For given $x \in U$ the affine forms $u \mapsto u(x)$ and $u \mapsto \widehat{R}_u^A(x)$ ($A \subset U$) are l.s.c. in the natural topology on $\mathcal{S}(U)$.*

Proof. Clearly, the map $u \mapsto \widehat{R}_u^A(x)$ is affine for fixed $x \in U$. Let (u_j) be a sequence in $\mathcal{S}(U)$ converging naturally to $u \in \mathcal{S}(U)$. For any index k we have

$$\widehat{\inf_{j \geq k} \widehat{R}_{u_j}^A}(x) \geq \widehat{R}_{\inf_{j \geq k} u_j}^A(x).$$

Either member of this inequality increases with k , and we get for $k \rightarrow \infty$

$$\liminf_k \widehat{R}_{u_k}^A(x) \geq \lim_k \widehat{\inf_k \widehat{R}_{u_k}^A}(x) \geq \widehat{R}_{\liminf_k u_k}^A(x) = \widehat{R}_u^A(x),$$

and so the map $u \mapsto \widehat{R}_u^A(x)$ is indeed l.s.c. on $\mathcal{S}(U)$. For the map $u \mapsto u(x)$ take $A = U$. \square

In the rest of the present section we denote by B a fixed compact base of $\mathcal{S}(U)$. As shown by Choquet (cf. [2, Corollary I.4.4]) the set $\text{Ext}(B)$ of extreme elements of B is a G_δ subset of B . On the other hand it follows by the fine Riesz decomposition theorem that every element of $\text{Ext}(B)$ is either a fine potential or an invariant function. We denote by $\text{Ext}_p(B)$ (resp. $\text{Ext}_i(B)$) the cone of all extreme fine potentials (resp. all extreme invariant functions) in B . According to the theorem on integral representation of fine potentials [22], any element of $\text{Ext}_p(B)$ has the form $\alpha G_U(\cdot, y)$, where α is a constant > 0 and $y \in U$.

Proposition 2.16. [12, Proposition 4.3]. *$\text{Ext}_p(B)$ and $\text{Ext}_i(B)$ are Borel subsets of B .*

When μ is a non-zero Radon measure on B there exists a unique element s of B such that

$$l(s) = \int_B l(u) d\mu(u)$$

for every continuous affine form $l : B \rightarrow [0, +\infty[$ on B (in other words, s is the barycenter of the probability measure $\frac{1}{\mu(B)}\mu$). For any l.s.c. affine form $\varphi : B \rightarrow [0, +\infty]$ there exists by [2, Corollary I.1.4] an increasing sequence of continuous affine forms on B which converges to φ , and hence

$$\varphi(s) = \int_B \varphi(u) d\mu(u).$$

In particular, for fixed $x \in U$ and $A \subset U$, the affine forms $u \mapsto u(x)$ and $u \mapsto \widehat{R}_u^A(x)$ are l.s.c. according to Corollary 2.15, and hence

$$s(x) = \int_B u(x) d\mu(u) \quad \text{and} \quad \widehat{R}_s^A(x) = \int_B \widehat{R}_u^A(x) d\mu(u).$$

The following theorem is a particular case of Choquet's theorem.

Theorem 2.17. [12, Théorème 4.1] *For any $s \in \mathcal{S}(U)$ there exists a unique Radon measure on B carried by $\text{Ext}(B)$ such that*

$$s(x) = \int_B u(x) d\mu(u), \quad x \in U.$$

The next two theorems are immediate consequences of [12, Théorème 4.5].

Theorem 2.18. *For any fine potential p on U there exists a unique Radon measure μ on B carried by $\text{Ext}_p(B)$ such that*

$$p(x) = \int_B q(x) d\mu(q), \quad x \in U.$$

Theorem 2.19. [12, Théorème 4.6] *For any invariant function $h \in \mathcal{S}(U)$ there exists a unique Radon measure μ on B carried by $\text{Ext}_i(B)$ such that*

$$h(x) = \int_B k(x) d\mu(k), \quad x \in U.$$

The following theorem is an immediate consequence of Theorems 2.18 and 2.19.

Theorem 2.20. *Let $A \subset \text{Ext}_p(B)$ (resp. $A \subset \text{Ext}_i(B)$), and let μ be a Radon measure on B . Then $\int_A u d\mu(u)$ is a fine potential (resp. an invariant function).*

Remark 2.21. In view of [16, Section 11.16] the set $\mathcal{H}_i(U)$ of all invariant functions on U is clearly a lower complete and conditionally upper complete sublattice of $\mathcal{S}(U)$ in the specific order. According to Lemma 2.2 the specific order on $\mathcal{H}_i(U)$ coincides with the pointwise order.

Remark 2.22. On a Euclidean domain the invariant functions are the same as the harmonic functions ≥ 0 . It is well known in view of Harnack's principle that the set of these functions is closed in $\mathcal{H}(U)$ with the natural topology (which coincides with the topology of R.-M. Hervé [25]). However, when U is just a regular fine domain in a Green space Ω , the set $\mathcal{H}_i(U)$ of invariant functions on U need not be closed in the induced natural topology on U , as shown by the following example: Let $y \in U$ be a Euclidean non-inner point of U , and let (y_k) be a sequence of points of $\Omega \setminus U$ which converges Euclidean to y . The sequence $(G_\Omega(\cdot, y_k)|_U)$ then converges naturally in $\mathcal{S}(U)$ to $G_\Omega(\cdot, y)|_U$, which does not belong to $\mathcal{H}_i(U)$ because its fine potential part $G_U(\cdot, y)$ is non-zero.

3. MARTIN COMPACTIFICATION OF U AND INTEGRAL REPRESENTATION IN $\mathcal{S}(U)$

We continue considering a regular fine domain U in a Greenian domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Let B be a compact base of the cone $\mathcal{S}(U)$ and let $\Phi : \mathcal{S}(U) \rightarrow [0, +\infty[$ be a continuous affine form such that

$$B = \{u \in \mathcal{S}(U) : \Phi(u) = 1\}.$$

Then $\Phi(u) > 0$ except at $u = 0$. Consider the mapping $\varphi : U \rightarrow B$ defined by

$$\varphi(y) = P_y = \frac{G_U(\cdot, y)}{\Phi(G_U(\cdot, y))}.$$

Note that φ is injective because $y = \{x \in U : G_U(x, y) = +\infty\} = \{x \in U : \varphi(y)(x) = +\infty\}$. We may therefore identify $y \in U$ with $\varphi(y) = P_y \in B$ and hence U with $\varphi(U)$.

We denote by \overline{U} the closure of U in B (with the natural topology), and write $\Delta(U) = \overline{U} \setminus U$. Then \overline{U} is compact in B and is called the Martin compactification of U , and $\Delta(U)$ is called the Martin boundary of U .

If B and B' are two bases of $\mathcal{S}(U)$ the Martin compactifications of U relative to B and to B' are clearly homeomorphic.

Throughout the rest of this article we fix the compact base B of the cone $\mathcal{S}(U)$ and the above continuous affine form $\Phi : \mathcal{S}(U) \rightarrow]0, +\infty[$ defining this base.

For any $Y \in \overline{U}$ consider the function $K(\cdot, Y) \in B \subset \mathcal{S}(U) \setminus \{0\}$ defined on U by $K(x, Y) = \varphi(Y)(x)$ if $Y \in U$ and $K(\cdot, Y) = Y$ if $Y \in \Delta(U)$. Clearly the map $Y \mapsto K(\cdot, Y)$ is a bijection of \overline{U} on B .

Definition 3.1. The function $K : U \times \overline{U} \rightarrow]0, +\infty]$ defined by $K(x, Y) = K(\cdot, Y)(x)$ is called the (fine) Riesz-Martin kernel for U , and its restriction to $U \times \Delta(U)$ is called the (fine) Martin kernel for U .

Proposition 3.2. *The Riesz-Martin kernel $K : U \times \overline{U} \rightarrow]0, +\infty]$ has the following properties, \overline{U} being given the natural topology:*

- (i) *For any $x \in U$, $K(x, \cdot)$ is l.s.c. on \overline{U} .*
- (ii) *For any $Y \in \overline{U}$, $K(\cdot, Y) \in \mathcal{S}(U)$ is finely continuous on U .*
- (iii) *K is l.s.c. on $U \times \overline{U}$ when U is given the fine topology and \overline{U} the natural topology.*

Proof. (i) follows from Corollary 2.15 applied to $u = K(\cdot, Y)$ while identifying $K(\cdot, Y)$ with Y .

(ii) is obvious.

(iii) Let $x_0 \in U$, $Z \in \overline{U}$, and let (V_j) be a fundamental system of open neighborhoods of Z in \overline{U} such that $V_{j+1} \subset V_j$ for any j . For a given constant

$c > 0$ consider the increasing sequence of functions

$$k_j := \inf_{Y \in V_j} K(., Y) \wedge c$$

and their finely l.s.c. regularizations $\widehat{k}_j \in \mathcal{S}(U)$. By the Brelot property, cf. [19], there exists a fine neighborhood H of x_0 in U such that H is compact (in the Euclidean topology) and that the restrictions of the functions $\widehat{k}_j \in \mathcal{S}(U)$ and of $K(., Z) \wedge c \in \mathcal{S}(U)$ to H are continuous on H (again with the induced Euclidean topology). By (i) we have pointwise on U

$$K(., Z) \wedge c = \liminf_{Y \rightarrow Z} K(., Y) \wedge c = \sup_j \inf_{Y \in V_j} K(., Y) \wedge c,$$

which quasi-everywhere and hence everywhere on U equals $\sup_j \widehat{\inf_{Y \in V_j} K(., Y) \wedge c}$. By Theorem 2.12 and Dini's theorem there exists for given $\varepsilon > 0$ an integer $j_0 > 0$ such that

$$K(., Z) \wedge c = \sup_j \widehat{\inf_{Y \in V_j} K(., Y) \wedge c} = \sup_j \widehat{k}_j < \widehat{k}_i + \varepsilon$$

on H for any $i \geq j_0$. For any fine neighborhood W of x_0 with $W \subset H$ we have

$$\begin{aligned} \inf_{x \in W, Y \in V_j} K(x, Y) \wedge c &= \inf_{x \in W} k_j(x) \geq \inf_{x \in W} \widehat{k}_j(x) \\ &\geq \inf_{x \in W} K(x, Z) \wedge c - \varepsilon \geq K(x_0, Z) \wedge c - 2\varepsilon \end{aligned}$$

for $j \geq j_0$. The assertion (iii) follows by letting $\varepsilon \rightarrow 0$ and next $c \rightarrow +\infty$. \square

Remark 3.3. The Riesz-Martin kernel K is in general not l.s.c. in the product of the induced natural topology on U and the natural topology on \overline{U} . Not even the function $K(., y) = G_U(., y)/\Phi(G_U(., y))$, or equivalently $G_U(., y)$ itself, is l.s.c. on U with the induced natural topology for fixed $y \in U$. For if the set $V := \{x \in U : G_U(x, y) > 1\}$ were open for every $y \in U$ then U would be a natural neighborhood of y for every $y \in U$, that is, U would be naturally open in \overline{U} . But that is in general not the case because $\overline{U} \setminus U = \Delta(U)$ need not be naturally compact, see Example 3.8 below.

Remark 3.4. A set $A \subset U$ is termed a Euclidean nearly Borel set if it differs by a polar set from a Euclidean Borel set. We denote by $\mathcal{B}(U)$, resp. $\mathcal{B}^*(U)$, the σ -algebra of all Euclidean Borel, resp. nearly Borel subsets of U . Every finely open set $V \subset U$ is Euclidean nearly Borel because its regularized $r(V)$ is a Euclidean F_σ -set and $r(V) \setminus V$ is polar. It follows that every open subset W of $U \times \overline{U}$, now with the fine topology on U (and of course the natural topology on \overline{U}), belongs to the σ -algebra $\mathcal{B}^*(U) \times \mathcal{B}(\overline{U})$ generated by all sets $A_1 \times A_2$ where $A_1 \in \mathcal{B}^*(U)$ and where $A_2 \in \mathcal{B}(\overline{U})$, that is, A_2 is a Borel subset of \overline{U} . In view of Proposition 3.2 (iii) every set $\{(x, Y) \in U \times \overline{U} : K(x, Y) > \alpha\}$ ($\alpha \in \mathbb{R}$)

is such an open set W and therefore belongs to $\mathcal{B}^*(U) \times \mathcal{B}(\overline{U})$. This means that the Riesz-Martin kernel K is measurable with respect to $\mathcal{B}^*(U) \times \mathcal{B}(\overline{U})$.

Definition 3.5. An invariant function $h \in \mathcal{S}(U)$ is termed *minimal* if it belongs to an extreme generator of the cone $\mathcal{S}(U)$.

Recall that $\text{Ext}(B)$ denotes the set of extreme points of B , and $\text{Ext}_i(B)$ the subset of minimal invariant functions in B .

Proposition 3.6. *For any point $Y \in \Delta(U)$, if the function $K(., Y)$ is an extreme point of B , then $K(., Y)$ is a minimal invariant function.*

Proof. By the Riesz decomposition of functions from $\mathcal{S}(U)$, $K(., Y)$ is either a minimal fine potential on U or else a minimal invariant function on U . In the former case it follows by the integral representation of fine potentials that $K(., Y)$ must be equal to P_y for some $y \in U$, which contradicts $Y \in \Delta(U)$. Thus $K(., Y)$ is indeed a minimal invariant function. The converse is obvious. \square

Definition 3.7. A point $Y \in \Delta(U)$ is termed minimal if the function $K(., Y)$ is minimal, that is, it belongs to an extreme generator of the cone $\mathcal{S}(U)$.

We denote by $\Delta_1(U)$ the set of all minimal points of $\Delta(U)$. Contrary to the case where U is Euclidean open in Ω , $\Delta(U)$ is in general not compact (in the natural topology), as shown by the following example.

Example 3.8. Let ω be a Hölder domain in \mathbb{R}^n ($n \geq 2$) such that ω is irregular with a single irregular boundary point z (for example a Lebesgue spine), and take $U = \omega \cup \{z\}$. According to [1, Theorems 1 and 3.1] the Euclidean boundary $\partial\omega$ of ω is contained in $\Delta(\omega)$. It follows that z belongs to the Euclidean closure of $\Delta(\omega) \setminus \{z\}$. But $\Delta(U) = \Delta(\omega) \setminus \{z\}$, where z is identified with P_z , and since $\Delta(\omega)$ is compact we infer that $\Delta(U)$ is noncompact.

However, we have the following

Proposition 3.9. $\Delta(U)$ is a G_δ of \overline{U} .

Proof. Let (B_l) be a sequence of open balls in \mathbb{R}^n with Euclidean closures $\overline{B_l} \subset \Omega$ and such that $\Omega = \bigcup_l B_l$. For integers $k, l > 0$ put

$$A_{kl} = \{y \in U : \Phi(G_U(., y)) \geq 1/k\} \cap \overline{B_l}.$$

The sets A_{kk} cover U because $G_U(., y) > 0$ and hence $\Phi(G_U(., y)) > 0$. We first show that each A_{kl} is compact in \overline{U} with the natural topology. Let (y_j) be a sequence of points of A_{kl} . After passing to a subsequence we may suppose that (y_j) converges Euclidean to a point $y \in \overline{B_l}$, and that the sequences $(G_U(., y_j))$

and $(\widehat{R}_{G_\Omega(\cdot, y_j)}^{\mathcal{C}U})$ (restricted to U) converge in $\mathcal{S}(U) \cup \{+\infty\}$. It follows that (with $G_\Omega(\cdot, y_j)$ restricted to U)

$$\Phi(G_\Omega(\cdot, y_j)) = \Phi(G_U(\cdot, y_j)) + \Phi(\widehat{R}_{G_\Omega(\cdot, y_j)}^{\mathcal{C}U}),$$

and hence by passing to the limit in $\mathcal{S}(U) \cup \{+\infty\}$ as $j \rightarrow +\infty$

$$(3.1) \quad \Phi(G_\Omega(\cdot, y)) = \lim_j \Phi(G_U(\cdot, y_j)) + \lim_j \Phi(\widehat{R}_{G_\Omega(\cdot, y_j)}^{\mathcal{C}U}).$$

On the other hand,

$$\widehat{R}_{G_\Omega(\cdot, y)}^{\mathcal{C}U} = \widehat{R}_{\liminf_j G_\Omega(\cdot, y_j)}^{\mathcal{C}U} \leq \liminf_j \widehat{R}_{G_\Omega(\cdot, y_j)}^{\mathcal{C}U}.$$

The restriction of the function $\widehat{R}_{G_\Omega(\cdot, y)}^{\mathcal{C}U}$ to U being invariant according to Lemma 2.4 it follows by Lemma 2.2 that $\widehat{R}_{G_\Omega(\cdot, y)}^{\mathcal{C}U} \preccurlyeq \liminf_j \widehat{R}_{G_\Omega(\cdot, y_j)}^{\mathcal{C}U}$ (after restriction to U here, and often in the rest of the proof), and hence

$$\Phi(\widehat{R}_{G_\Omega(\cdot, y)}^{\mathcal{C}U}) \leq \Phi(\liminf_j \widehat{R}_{G_\Omega(\cdot, y_j)}^{\mathcal{C}U}) = \lim_j \Phi(\widehat{R}_{G_\Omega(\cdot, y_j)}^{\mathcal{C}U}).$$

We infer from (3.1) by the definition of A_{kl} that

$$\Phi(G_\Omega(\cdot, y)) \geq \frac{1}{k} + \Phi(\widehat{R}_{G_\Omega(\cdot, y)}^{\mathcal{C}U}),$$

and consequently $y \in U$ and $\Phi(G_U(\cdot, y)) \geq 1/k$, whence $y \in A_{kl}$. Now put $s = \lim_j G_U(\cdot, y_j)$. We have $s > 0$ because $y_j \in A_{kl} \subset U$ and hence

$$\Phi(s) = \lim_j \Phi(G_U(\cdot, y_j)) \geq 1/k.$$

Hence

$$\varphi(y_j) = \frac{G_U(\cdot, y_j)}{\Phi(G_U(\cdot, y_j))} \rightarrow \frac{s}{\Phi(s)} \in \mathcal{S}(U).$$

This shows that A_{kl} is compact in the natural topology on \overline{U} , and consequently $\Delta(U) = \overline{U} \setminus U = \bigcap_{kl} (\overline{U} \setminus A_{kl})$ is indeed a G_δ in \overline{U} . \square

Corollary 3.10. $\Delta_1(U)$ is a G_δ of \overline{U} .

Proof. Because $\Delta_1(U) = \Delta(U) \cap \text{Ext}_i(B)$ it suffices according to Proposition 3.9 to use the well-known fact that $\text{Ext}(B)$ is a G_δ of B . \square

The proof of Proposition 3.9 also establishes

Corollary 3.11. For any integers $k, l > 0$ the set $C_{kl} := \{P_y : y \in A_{kl}\}$ is compact in B , and the mapping $\varphi : y \mapsto P_y$ is a homeomorphism of A_{kl} onto C_{kl} .

Corollary 3.12. Let $K \subset U$ be compact in the Euclidean topology. Then the set $C_K := \{P_y : y \in K\}$ is a (natural) K_σ in B , and so is therefore $\text{Ext}_p(B) = \{P_y : y \in U\}$.

Proof. Since $C_K = \bigcup_k \varphi(K \cap A_k)$ the assertion follows by the preceding corollary. \square

Proposition 3.13. *Let $u \in \mathcal{S}(U)$ and let $A \subset \overline{A} \subset U$, where \overline{A} denotes the closure of A in \overline{U} . The measure μ on B carried by the extreme points of B and representing \widehat{R}_u^A is then carried by \overline{A} .*

Proof. Let p be a finite fine potential > 0 on U . For any pair (k, l) and any integer $j > 0$ the function $\widehat{R}_{u \wedge j p}^A$ is a fine potential on U and finely harmonic on $U \setminus \overline{A}$ by Lemma 2.4. The measure μ_j on B carried by $\text{Ext}(B)$ and representing $\widehat{R}_{u \wedge j p}^A$ is carried by \overline{A} . The sequence of probability measures $\frac{1}{\Phi(\widehat{R}_{u \wedge j p}^A)} \mu_j$ has a subsequence (μ_{j_k}) which converges to a probability measure μ on B carried by \overline{A} . We thus have

$$\widehat{R}_u^A = \lim_{k \rightarrow \infty} \widehat{R}_{u \wedge j_k p}^A = \Phi(\widehat{R}_u^A) \lim_{k \rightarrow \infty} \int q d\mu_{j_k}(q) = \Phi(\widehat{R}_u^A) \int q d\mu(q).$$

The assertion now follows by the fact that $\overline{A} \subset U \subset \text{Ext}(B)$ and from the uniqueness of the integral representation in Choquet's theorem. \square

Recall from the beginning of the present section the continuous affine form $\Phi \geq 0$ on $\mathcal{S}(U)$ such that the chosen compact base B of the cone $\mathcal{S}(U)$ consists of all $u \in \mathcal{S}(U)$ with $\Phi(u) = 1$. Also consider the compact sets $A_{kl} \subset U$ in the proof of Proposition 3.9. Cover Ω by a sequence of Euclidean open balls B_k with closures \overline{B}_k contained in Ω .

Lemma 3.14. (a) *The mapping $U \ni y \mapsto G_U(., y) \in \mathcal{S}(U)$ is continuous from U with the fine topology into $\mathcal{S}(U)$ with the natural topology.*

(b) *The function $U \ni y \mapsto \Phi(G_U(., y)) \in]0, +\infty[$ is finely continuous.*

(c) *The sets*

$$V_k = \{y \in U : \Phi(G_U(., y)) > 1/k\} \cap B_k$$

form a countable cover of U by finely open sets which are relatively naturally compact in U .

Proof. (a) Consider a net $(y_\alpha)_{\alpha \in I}$ on U converging finely to some $y \in U$, in other words $s(y_\alpha) \rightarrow s(y)$ for every $s \in \mathcal{S}(U)$. Taking $s = G_U(x, .)$ with $x \in U$ we have in particular $G_U(x, y_\alpha) \rightarrow G_U(x, y)$ for every $x \in U$. Since $G_U(., y_\alpha) \in \mathcal{S}(U) \cup \{+\infty\}$, which is naturally compact by Theorem 2.12, we may assume that $G_U(., y_\alpha) \rightarrow z$ naturally for some $z \in \mathcal{S}(U) \cup \{+\infty\}$. Furthermore, we then have

$$z = \lim_{\alpha} \widehat{\inf} G_U(., y_\alpha) = G_U(., y),$$

where the latter equality holds first q.e., and next everywhere on U by fine continuity of z and $G_U(., y)$. Thus $G_U(., y_\alpha) \rightarrow G_U(., y)$, which proves assertion (a).

(b) It follows from (a) that this function is finely continuous on U because the mapping $\Phi : \mathcal{S}(U) \rightarrow [0, +\infty[$ is (naturally) continuous, as recalled above.

(c) According to (b), each V_k is finely open along with $U \cap B_k$. By the proof of Proposition 3.9 we have $V_k \subset A_{kk} \subset U$ with A_{kk} naturally compact. Clearly, $\bigcup_k V_k = U$. \square

Corollary 3.15. *The mapping $\varphi : U \ni y \mapsto P_y = K(., y) \in \mathcal{S}(U)$ is continuous from U with the fine topology to $\mathcal{S}(U) \cup \{+\infty\}$ with the natural topology. In other words, the fine topology on U is finer than the topology on U induced by the natural topology on \overline{U} .*

Proof. Recall that $P_y = G_U(., y)/\Phi(G_U(., y))$ for $y \in U$. For any net (y_α) on U converging finely to $y \in U$ we obtain from (a) and (b) in the above lemma

$$\liminf_{\alpha} \widehat{P}_{y_\alpha} = \frac{\liminf_{\alpha} G_U(., y_\alpha)}{\Phi(G_U(., y))} = \frac{G_U(., y)}{\Phi(G_U(., y))} = P_y,$$

first quasi-everywhere, and next everywhere in U by fine continuity. \square

Theorem 3.16. *Every extreme element of the base B of the cone $\mathcal{S}(U)$ belongs to \overline{U} . In particular, any extreme invariant function h in B has the form $h = K(., Y)$ where $Y \in \Delta_1(U)$.*

Proof. Let p be an extreme element of B . By Riesz decomposition, either p is the fine potential of a measure supported by a single point $y \in U$, or else p is an invariant function on U . In the former case we have $p = P_y$ and hence $p \in U$. In the latter case it follows by the proof of Proposition 3.9 that there exists an increasing sequence of compact subsets (K_j) of \overline{U} (of the form A_{kl}) such that $\bigcup_j K_j = U$. For each j , $\widehat{R}_p^{K_j}$ is a fine potential on U . In fact, by Proposition 3.13, the measure μ on B carried by $\text{Ext}(B)$ and representing $\widehat{R}_p^{K_j}$ is carried by K_j , and hence

$$\widehat{R}_p^{K_j} = \int P_y d\mu(y) = \int G_U(., y) d\nu(y),$$

is a fine potential on U , the measure ν on B being well defined by $d\nu(y) = \frac{1}{\Phi(G_U(., y))} d\mu(y)$, cf. Corollary 3.15 (b). For any j there exists a Radon measure μ_j on B such that $\widehat{R}_p^{K_j} = \int q d\mu_j(q)$. The measure μ_j is carried by \overline{U} . Because p is invariant it follows by Lemma 2.2 that the sequence $\widehat{R}_p^{K_j}$ increases specifically to p as $j \rightarrow \infty$, and the sequence (μ_j) is therefore increasing. Consequently, $\int d\mu_j = \Phi(\widehat{R}_p^{K_j}) \rightarrow \Phi(p)$ as $j \rightarrow \infty$, and the sequence (μ_j) converges vaguely to a measure μ on \overline{U} . It follows that $p = \int_{\overline{U}} q d\mu(q)$, and since p is extreme we conclude that $p \in \overline{U}$ because μ must be carried by a single point. \square

Corollary 3.17. $\text{Ext}(B) = U \cup \Delta_1(U)$.

Proof. Every extreme element of B is either the fine potential of a measure supported by a single point $y \in U$, hence of the form P_y , or else a minimal invariant functions, hence of the form $K(., Y)$ with $Y \in \Delta_1(U)$, according to Proposition 3.6. This establishes the inclusion $\text{Ext}(B) \subset U \cup \Delta_1(U)$. The opposite inclusion is evident. \square

Theorem 3.18. *For any invariant function h on U there exists a unique Radon measure μ on \overline{U} carried by $\Delta_1(U)$ such that*

$$h(x) = \int_{\Delta(U)} K(x, Y) d\mu(Y), \quad x \in U.$$

Proof. The theorem follows immediately from Theorem 2.18 and Corollaries 3.11 and 3.17. \square

Proposition 3.19. *Let $u \in \mathcal{S}(U)$ and let V be a finely open Borel subset of U . Let μ be the measure on B carried by $\text{Ext}(B)$ and representing u . Then $\mu(V) = 0$ if and only if the restriction $u|_V$ is invariant.*

Proof. Write $u = p + h$ with $p \in \mathcal{P}(U)$ and h invariant on U . Let λ and ν be the measures on $\text{Ext}(B)$ representing p and h , respectively. Then $\mu = \lambda + \nu$ with $\nu(U) = 0$ according to Corollary 3.17 or 2.19 and Theorem 3.18. Writing $\Phi(G_U \mu) = \alpha$ we have by [21, Lemma 2.6]

$$\alpha p = G_U \lambda = G_V \lambda|_V + \widehat{R}_{G_U \lambda}^{U \setminus V} \preceq \alpha u \quad \text{on } V,$$

and hence $G_V \lambda|_V \preceq \alpha u$ on V . If $u|_V$ is invariant the fine potential $G_V \lambda|_V$ must therefore be 0, whence $\lambda(V) = 0$ and finally $\mu(V) = \lambda(V) + \nu(V) = 0$ because $\nu(V) \leq \mu(V) = 0$. Conversely, suppose that $\mu(V) = 0$ and hence $\lambda(V) = 0$. In the above display $\widehat{R}_{G_U \lambda}^{U \setminus V}$ is invariant on V by Lemma 2.4, and so is therefore p . It follows that $u = p + h$ is invariant on V , h being invariant on U and hence on V by Theorem 2.6 (a). \square

For any (positive) Borel measure μ on \overline{U} define a function $K\mu : U \rightarrow [0, +\infty]$ by

$$K\mu = \int K(., Y) d\mu(Y), \quad x \in U.$$

This integral exists in view of Proposition 3.2 (i).

Theorem 3.20. 1. *For any Borel measure μ on \overline{U} , $K\mu$ is finely hyperharmonic on U , that is, $K\mu \in \mathcal{S}(U) \cup \{+\infty\}$.*

2. *Every function $u \in \mathcal{S}(U)$ has a unique integral representation $u = K\mu$ in terms of a Borel measure μ on $U \cup \Delta_1(U)$.*

Proof. 1. The kernel K is measurable with respect to the product σ -algebra $\mathcal{B}^*(U) \times \mathcal{B}(\overline{U})$ in view of the conclusion in Remark 3.4. It follows that the function $K\mu$ is nearly Euclidean Borel measurable on U . We begin by showing

that $K\mu$ is nearly finely hyperharmonic, cf. [16, Definition 11.1]. Let $V \subset U$ be finely open with $\tilde{V} \subset U$. For any $x \in V$ the swept measure $\varepsilon_x^{\Omega \setminus V}$ is carried by the fine boundary $\partial_f V \subset U$ and does not charge any polar set. Hence $\varepsilon_x^{\Omega \setminus V}$ may be regarded as a measure on the σ -algebra $\mathcal{B}^*(U)$ of all Euclidean nearly Borel subsets of U , cf. again Remark 3.4. Altogether, Fubini's theorem applies, and we obtain for any $x \in U$:

$$\begin{aligned} \widehat{R}_{K\mu}^{U \setminus V}(x) &= \int_U K\mu(y) d\varepsilon_x^{\Omega \setminus V}(y) \\ &= \int_{\overline{U}} \left(\int_U K(y, Y) d\varepsilon_x^{\Omega \setminus V}(y) \right) d\mu(Y) \\ &\leq \int_{\overline{U}} K(x, Y) d\mu(Y) = K\mu(x), \end{aligned}$$

the inequality because $K(\cdot, Y) \in \mathcal{S}(U)$, cf. [16, Definition 8.1]. Thus $K\mu$ is nearly finely hyperharmonic on U . To show that $K\mu$ is actually finely hyperharmonic we shall prove that the finely l.s.c. envelope $\widehat{K\mu}$ of $K\mu$ equals $K\mu$, cf. [16, Lemma 11.2 and Definition 8.4] according to which

$$\widehat{K\mu}(x) = \sup_{V \in \mathcal{V}} \int_U K\mu d\varepsilon_x^{\Omega \setminus V},$$

where \mathcal{V} denotes the lower directed family of all finely open sets $V \subset U$ of Euclidean compact closure in Ω contained in U . For each $V \in \mathcal{V}$ and $x \in V$ we have, again by Fubini in view of Remark 3.4,

$$\begin{aligned} \int_U K\mu(y) d\varepsilon_x^{\Omega \setminus V}(y) &= \int_{\overline{U}} \left(\int_U K(y, Y) d\varepsilon_x^{\Omega \setminus V}(y) \right) d\mu(Y) \\ &= \int_{\overline{U}} \widehat{R}_{K(\cdot, Y)}^{U \setminus V}(x) d\mu(Y). \end{aligned}$$

Taking supremum over all $V \in \mathcal{V}$ leads to $\widehat{K\mu} = K\mu$ as desired. In fact, the increasing net of finely superharmonic functions $(\widehat{R}_{K(\cdot, Y)}^{U \setminus V})_{V \in \mathcal{V}}$ admits an increasing subsequence with the same pointwise supremum, by [16, Remark (p. 91)], and this supremum equals $\widehat{R}_{K(\cdot, Y)}^{U \setminus \{x\}} = \widehat{R}_{K(\cdot, Y)}^U = K(\cdot, Y)$. It follows that

$$\begin{aligned} \int_U K\mu d\varepsilon_x^{U \setminus V} &= \int_{\overline{U}} \widehat{R}_{K(\cdot, Y)}^{U \setminus V}(x) d\mu(Y) \\ &\nearrow \int_{\overline{U}} K(x, Y) d\mu(Y) = K\mu(x) \end{aligned}$$

according to [16, Theorem 11.12]. We conclude that $\widehat{K\mu} = K\mu$, and so $K\mu$ is indeed finely hyperharmonic on U .

2. As noted in [12, Theorem 4.1] this follows immediately from Choquet's integral representation theorem applied to the cone $\mathcal{S}(U)$ with the base B . \square

Lemma 3.21. *For any set $A \subset U$ and any Radon measure μ on \overline{U} we have $\widehat{R}_{K\mu}^A = \int \widehat{R}_{K(\cdot, Y)}^A d\mu(Y)$.*

Proof. As in the proof of Lemma 2.3, $\widehat{R}_{K\mu}^A$ and $\widehat{R}_{K(\cdot, Y)}^A$ remain unchanged when A is replaced by $b(A) \cap U$, and we may therefore assume that $A = b(A) \cap U$, whence $\widehat{R}_{K\mu}^A = R_{K\mu}^A$ and $\widehat{R}_{K(\cdot, Y)}^A = R_{K(\cdot, Y)}^A$. As in the proof of Theorem 3.20 the kernel K on $U \times \overline{U}$ is measurable with respect to $\mathcal{B}^*(U) \times \mathcal{B}(\overline{U})$. Furthermore, $K\mu$ is finely hyperharmonic on U , and we have by Lemma 2.3 and Fubini for any $x \in U$

$$\begin{aligned} \widehat{R}_{K\mu}^A(x) &= R_{K\mu}^A(x) = \int_U K\mu d\varepsilon_x^{A \cup (\Omega \setminus U)} \\ &= \int_{\overline{U}} d\mu(Y) \int_U K(\cdot, Y) d\varepsilon_x^{A \cup (\Omega \setminus U)} = \int_{\overline{U}} R_{K(\cdot, Y)}^A(x) d\mu(Y) \\ &= \int_{\overline{U}} \widehat{R}_{K(\cdot, Y)}^A(x) d\mu(Y). \end{aligned}$$

\square

Corollary 3.22. *Let μ be a Borel measure on $\Delta_1(U)$ and let $h = K\mu$. If h is finite q.e. then $h \in \mathcal{S}(U)$ and h is invariant.*

Proof. Let V be a regular finely open set such that $\widetilde{V} \subset U$. For any $x \in V$ the measure $\varepsilon_x^{\Omega \setminus V}$ is carried by $\partial_f V \subset \widetilde{V} \subset U$ and does not charge any polar set. This measure may therefore be regarded as a measure on the σ -algebra $\mathcal{B}^*(U)$ of all nearly Borel subsets of U , cf. Remark 3.4, where it is also shown that K is measurable with respect to the product σ -algebra $\mathcal{B}^*(U) \times \mathcal{B}(\overline{U})$. Supposing that $h < +\infty$ q.e. on U we may therefore apply Fubini's theorem and Lemma 2.3 to obtain

$$\begin{aligned} \widehat{R}_h^{U \setminus V}(x) &= R_h^{U \setminus V}(x) = \int_U h(y) d\varepsilon_x^{\Omega \setminus V}(y) \\ &= \int_{\Delta_1(U)} \left(\int_U K(y, Y) d\varepsilon_x^{\Omega \setminus V}(y) \right) d\mu(Y) \\ &= \int_{\Delta_1(U)} R_{K(\cdot, Y)}^{U \setminus V}(x) d\mu(Y) = \int_{\Delta_1(U)} \widehat{R}_{K(\cdot, Y)}^{U \setminus V}(x) d\mu(Y) \\ &= \int_{\Delta_1(U)} K(x, Y) d\mu(Y) = h(x). \end{aligned}$$

In the remaining case $x \in U \setminus V$ the resulting equation $\widehat{R}_h^{U \setminus V}(x) = h(x)$ holds because x belongs to $U \setminus V$ which is a base relative to U . According to [21,

Theorem 4.4] there is a countable cover of U by sets like the above set V , and since $\widehat{R}_h^{U \setminus V}(x) = h(x)$ for each such set, we conclude from the quoted theorem that indeed h is invariant. \square

Remark 3.23. The finiteness condition on h in Corollary 3.22 is equivalent with $h \neq +\infty$, that is $h \in \mathcal{S}(U)$.

Corollary 3.24. *For any finite measure μ on $\Delta_1(U)$ the function $h = \int_{\Delta_1(U)} K(., Y) d\mu(Y)$ is an invariant function.*

Proof. Let ν be the measure on B defined by $\nu(A) = \mu(\Delta_1(U) \cap A)$ for any Borel set $A \subset B$. Then ν is a finite measure on B , and we may suppose that $|\nu| = 1$. Let $h \in B$ be the barycenter of ν . Then $h = \int_{\Delta_1(U)} K(., Y) d\mu(Y)$, and hence h is invariant according to Theorem 2.18 and Corollary 3.17. \square

Corollary 3.25. *Let μ be a Borel measure on $U \cup \Delta_1(U)$. A function $u = K\mu \neq +\infty$ is a fine potential, resp. an invariant function, if and only μ is carried by U , resp. by $\Delta_1(U)$.*

Proof. For $y \in U$ write $\alpha(y) := \Phi(G_U(., y))$. If μ is carried by U then $K\mu(y) = G_U(\alpha(y)^{-1}\mu)$ is a fine potential on U . Conversely, if u is a fine potential on U then there is a measure ν on U such that $u(y) = G_U\nu(y) = K(\alpha(y)\nu)$. On the other hand, if μ is carried by $\Delta_1(U)$ then $u = p + h$ with $p = K\mu$ for some μ on U and $h = K\lambda$ for some λ carried by $\Delta_1(U)$. By uniqueness in Theorem 3.20, $\mu = \nu + \lambda$, where ν is carried by U and hence $\nu = 0$, so that $K\mu = K\lambda$ is invariant according to Corollary 3.22. Conversely, if $K\mu$ is invariant then $u = p + h = K\nu + K\lambda$ as above, and here h is invariant according to Corollary 3.24. It follows that $p = 0$, that is $\nu = 0$, and so $\mu = \lambda$ is carried by $\Delta_1(U)$. \square

Corollary 3.26. *Let $u \in \mathcal{S}(U)$ and $A \subset \overline{A} \subset U$, where \overline{A} denotes the closure of A in \overline{U} . Then \widehat{R}_u^A is a fine potential on U .*

Proof. According to Proposition 3.13 the measure μ on $U \cup \Delta_1(U)$ representing \widehat{R}_u^A is carried by \overline{A} . It follows by the preceding corollary that \widehat{R}_u^A is a fine potential. \square

We close this section with the following characterizations of the invariant functions and the fine potentials on U . These characterizations are analogous (but only partly comparable) to [21, Theorems 4.4 and 4.5], respectively, where the present condition $\overline{V} \subset U$ was replaced by the weaker condition $\widetilde{V} \subset U$.

Theorem 3.27. *Let $u \in \mathcal{S}(U)$. Then u is invariant if and only if $\widehat{R}_u^{U \setminus V} = u$ for any regular finely open set $V \subset U$ such that the closure \overline{V} of V in the natural topology on B is contained in U .*

Proof. Suppose that u is invariant. For any V as stated we have $u \leq \widehat{R}_u^V + \widehat{R}_u^{U \setminus V}$. By the Riesz decomposition property [16, p. 129] there are functions $u_1, u_2 \in \mathcal{S}(U)$ such that $u = u_1 + u_2$ with $u_1 \leq \widehat{R}_u^V$ and $u_2 \leq \widehat{R}_u^{U \setminus V}$. But \widehat{R}_u^V is a fine potential by Corollary 3.26, and so is therefore u_1 . It follows that $u_1 = 0$ because $u_1 \preceq u$ and u is invariant. Consequently, $u \leq \widehat{R}_u^{U \setminus V}$, and so indeed $\widehat{R}_u^{U \setminus V} = u$. Conversely, suppose that $\widehat{R}_u^{U \setminus V} = u$ for any regular finely open (and finely connected, if we like) set $V \subset U$ with $\overline{V} \subset U$. Let μ be the (finite) measure on $U \cup \Delta_1(U)$ which represents u . Then

$$u = \int_U P_y d\mu(y) + \int_{\Delta_1(U)} K(., Y) d\mu(Y).$$

For any regular fine domain $V \subset U$ with $\overline{V} \subset U$ we have by hypothesis and by Lemma 3.21

$$u = \widehat{R}_u^{U \setminus V} = \int_U \widehat{R}_{P_y}^{U \setminus V} d\mu(y) + \int_{\Delta_1(U)} \widehat{R}_{K(., Y)}^{U \setminus V} d\mu(Y),$$

and hence

$$\int_U (P_y - \widehat{R}_{P_y}^{U \setminus V}) d\mu(y) = 0.$$

Since V is regular and since $P_y - \widehat{R}_{P_y}^{U \setminus V} = G_V(., y) / \Phi(G_U(., y)) > 0$ on V in view of [21, Lemma 2.6], it follows that $\mu(V) = 0$. This implies by Lemma 3.14 (c) (together with the fact that a finely open set has only countably many fine components) that μ is carried by $\Delta_1(U)$. Consequently, u is invariant according to Theorem 2.20. \square

Corollary 3.28. *Let $u \in \mathcal{S}(U)$. Then u is a fine potential if and only if $\widehat{\inf}_j \widehat{R}_u^{U \setminus V_j} = 0$ for some, and hence any, cover of U by an increasing sequence (V_j) of regular finely open sets such that $\overline{V_j} \subset V_{j+1} \subset U$ for every j .*

Proof. Suppose first that u is a fine potential, and consider any cover (V_j) of U as stated. Denote $v = \widehat{\inf}_j \widehat{R}_u^{U \setminus V_j}$, which is likewise a fine potential. For each index k , $\widehat{R}_u^{U \setminus V_j}$ is invariant on V_k for any $j \geq k$ according to Lemma 2.4. It follows by Theorem 2.6 (c) that v likewise is invariant on V_k . By varying k we see from Theorem 2.6 (b) applied to the regular finely open sets V_k that v is invariant on $\bigcup_k V_k = U$. Consequently $v = 0$. Conversely, suppose that $\widehat{\inf}_j \widehat{R}_u^{U \setminus V_j} = 0$ for some cover (V_j) as stated in the corollary. We have $u = p + h$, where p is a fine potential and h is an invariant function. By the preceding theorem, $h = \widehat{R}_h^{U \setminus V_j} \leq \widehat{R}_u^{U \setminus V_j}$ for every j , and hence $h = 0$, showing that u is a fine potential. \square

4. THE FATOU-NAÏM-DOOB THEOREM FOR FINELY SUPERHARMONIC FUNCTIONS

As mentioned in the Introduction, this section is inspired by the axiomatic approach to the Fatou-Naïm-Doob theorem given in [31]. These axioms are, however, only partially fulfilled in our setting. In particular, our invariant functions, which play the role of positive harmonic functions, may take the value $+\infty$. We therefore choose to give the proof of the Fatou-Naïm-Doob theorem without reference to the proofs in [31].

We continue considering a regular fine domain U in a Greenian domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Recall that $\mathcal{P}(U)$ denotes the band in $\mathcal{S}(U)$ consisting of all fine potentials on U , and that the orthogonal band $\mathcal{P}(U)^\perp = \mathcal{H}_i(U)$ relative to $\mathcal{S}(U)$ consists of all invariant functions h on U ; these are characterized within $\mathcal{S}(U)$ by their integral representation $h = K\mu$, that is,

$$h(x) = K\mu(x) = \int K(x, Y) d\mu(Y)$$

in terms of a unique measure μ on \overline{U} carried by the minimal Martin boundary $\Delta_1(U)$ (briefly: a measure on $\Delta_1(U)$), see Theorem 3.20 and Corollary 3.25. In the present section we shall not consider the whole Riesz-Martin space \overline{U} and the full Riesz-Martin kernel $K : U \times \overline{U} \rightarrow]0, +\infty]$ (Definition 3.1), but only the Martin boundary $\Delta(U)$ and the Martin kernel, the restriction of the Riesz-Martin kernel to $U \times \Delta(U)$, and K will henceforth stand for this restriction. It is understood that U and $\Delta(U)$ are given the natural topology (induced by the natural topology on the Riesz-Martin space \overline{U}). In particular, by Lemma 3.14(c), U is a K_σ subset of \overline{U} , and we know that $\Delta(U)$ and the minimal Martin boundary $\Delta_1(U)$ are G_δ subsets of \overline{U} (Proposition 3.9 and Corollary 3.10). We shall need the following two preparatory lemmas.

Lemma 4.1. (a) *If $h_1, h_2 \in \mathcal{H}_i(U)$, $p \in \mathcal{P}(U)$, and if $h_1 \leq h_2 + p$, then $h_1 \preccurlyeq h_2$ and $h_2 - h_1 \in \mathcal{H}_i(U)$.*

(b) *If (h_j) is an increasing sequence of functions $h_j \in \mathcal{H}_i(U)$ majorized by some $u \in \mathcal{S}(U)$ then $\sup_j h_j \in \mathcal{H}_i(U)$.*

It is understood in (a) (and similarly elsewhere) that $h_2 - h_1$ is defined to be the extension by fine continuity from $\{x \in U : h_1(x) < +\infty\}$ to U , cf. [16, Theorem 9.14]. (Equivalently, $h_2 - h_1$ is well defined because $\mathcal{S}(U)$ is an H -cone, as noted after the proof of Theorem 2.7.) For any set $A \subset U$, R_u^A and \widehat{R}_u^A are understood as reduction and sweeping of a function u on U relative to U , whereas ε_x^A stands for sweeping of ε_x on A relative to all of Ω .

Proof. (a) There exists $s \in \mathcal{S}(U)$ such that $h_1 + s = h_2 + p$. We have $s = h + q$ with $h \in \mathcal{H}_i(U)$ and $q \in \mathcal{P}(U)$. It follows that $(h_1 + h) + q = h_2 + p$ with $h_1 + h \in \mathcal{H}_i(U)$, and hence $h_1 + h = h_2$ (and $q = p$).

(b) According to (a) the sequence (h_j) is even specifically increasing. Because $\sup_j u_j \in \mathcal{S}(U)$ along with u we have $\sup_j u_j = \gamma_j h_j$ ([16, (b), p. 132], which belongs to the band $\mathcal{H}_i(U)$ along with each h_j . \square

Lemma 4.2. *For any set $E \subset U$ and any $Y \in \Delta_1(U)$ we have $\widehat{R}_{K(.,Y)}^E \neq K(., Y)$ if and only if $\widehat{R}_{K(.,Y)}^E \in \mathcal{P}(U)$.*

Proof. Note that, for any $u \in \mathcal{S}(U)$ and $E \subset U$, we have $R_u^E \neq u \iff \widehat{R}_u^E \neq u$. Proceeding much as in [31, proof of G)], see also [24], we first suppose that $\widehat{R}_{K(.,Y)}^E \neq K(., Y)$. Since $R_{K(.,Y)}^E \leq K(., Y)$ there exists $x_0 \in U$ with $R_{K(.,Y)}^E(x_0) < K(x_0, Y)$. Thus there exists $u \in \mathcal{S}(U)$ such that $u \geq K(., Y)$ on E and $u(x_0) < K(x_0, Y)$. Replacing u by $u \wedge K(., Y) \in \mathcal{S}(U)$ we arrange that $u \leq K(., Y)$ on all of U . Writing $u = q + h$ with $q \in \mathcal{P}(U)$ and $h \in \mathcal{U}_i(U)$ we have $h \leq u \leq K(., Y)$ on U . Because $K(., Y) \in \mathcal{H}_i$ is minimal invariant and $h \preceq K(., Y)$ by Lemma 4.1 (with $p = 0$), there exists $\alpha \in [0, 1]$ such that $h = \alpha K(., Y)$, and here $\alpha < 1$ since $h(x_0) \leq u(x_0) < K(x_0, Y)$. On E we have $q = u - h = K(., Y) - h = (1 - \alpha)K(., Y)$, and hence

$$K(., Y) = p := \frac{1}{1 - \alpha} q \quad \text{on } E.$$

Thus $p \in \mathcal{P}(U)$ and $\widehat{R}_{K(.,Y)}^E = \widehat{R}_p^E \leq p$ on U , so indeed $\widehat{R}_{K(.,Y)}^E \in \mathcal{P}(U)$. Conversely, suppose that $\widehat{R}_{K(.,Y)}^E = K(., Y)$ and (by contradiction) that $\widehat{R}_{K(.,Y)}^E \in \mathcal{P}(U)$. Being thus both a fine potential and invariant, $K(., Y)$ must equal 0, which is false. \square

Definition 4.3. A set $E \subset U$ is said to be minimal-thin at a point $Y \in \Delta_1(U)$ if $\widehat{R}_{K(.,Y)}^E \neq K(., Y)$, or equivalently (by the preceding lemma) if $\widehat{R}_{K(.,Y)}^E$ is a fine potential on U .

Corollary 4.4. *For any $Y \in \Delta_1(U)$ the sets $W \subset U$ for which $U \setminus W$ is minimal-thin at Y form a filter on U .*

This follows from Lemma 4.2 which easily implies that for any $W_1, W_2 \subset U$ such that $\widehat{R}_{K(.,Y)}^{U \setminus W_i} \neq K(., Y)$ for $i = 1, 2$, we have $\widehat{R}_{K(.,Y)}^{U \setminus (W_1 \cup W_2)} \neq K(., Y)$.

The filter from Corollary 4.4 is called the minimal-fine filter at Y and will be denoted by $\mathcal{F}(Y)$. A limit along that filter will be called a minimal-fine limit and will be denoted by $\lim_{\mathcal{F}(Y)}$.

For any two functions $u, v \in \mathcal{S}(U)$ with $v \neq 0$ the quotient u/v is assigned some arbitrary value, say 0, on the polar set of points at which both functions take the value $+\infty$. The choice of such a value does not affect a possible minimal-fine limit of u/v at a point $Y \in \Delta_1(U)$ because every polar set E clearly is minimal-thin at any point of $\Delta_1(U)$.

For any two measures μ, ν on a measurable space we denote by $d\nu/d\mu$ the Radon-Nikodým derivative of the absolutely continuous component of ν relative to that of μ , cf. e.g. [11, p. 773], [4, p. 305f].

For any function $u \in \mathcal{S}(U)$ with Riesz decomposition $u = p + h$, where $p \in \mathcal{P}(U)$ and $h \in \mathcal{H}_i(U)$, we denote by μ_u the unique measure on $\Delta_1(U)$ which represents the invariant part h of u , that is, $h = \int K(., Y) d\mu_u(Y)$.

We may now formulate the Fatou-Naïm-Doob theorem in the present setting of finely superharmonic functions on a regular fine domain U . It clearly contains the classical Fatou-Naïm-Doob theorem for which we refer to [11, 1.XII.19], [4, 9.4].

Theorem 4.5. *Let $u, v \in \mathcal{S}(U)$, where $v \neq 0$. Then u/v has minimal-fine limit $d\mu_u/d\mu_v$ at μ_v -a.e. point Y of $\Delta_1(U)$.*

For the proof of Theorem 4.5 we begin by establishing the following important particular case, cf. [31, Theorem 1.2].

Proposition 4.6. ([31].) *Let $u \in \mathcal{S}(U)$ and $h \in \mathcal{H}_i(U) \setminus \{0\}$, and suppose that $u \wedge h \in \mathcal{P}(U)$. Then u/h has minimal-fine limit $d\mu_u/d\mu_h = 0$ at μ_h -a.e. point Y of $\Delta_1(U)$.*

Proof. Write $u \wedge h = p$. Given $\alpha \in]0, 1[$, consider any point $Y \in \Delta_1(U)$ such that $\limsup_{\mathcal{F}(Y)} \frac{u}{h} > \alpha$. Then $\{u \leq \alpha h\} \notin \mathcal{F}(Y)$, that is, $\{u > \alpha h\}$ is not minimal-thin at Y :

$$(4.1) \quad \widehat{R}_{K(., Y)}^{\{u > \alpha h\}} = K(., Y).$$

It follows that (always with Y ranging over $\Delta_1(U)$)

$$(4.2) \quad \{Y : \limsup_{\mathcal{F}(Y)} \frac{u}{h} > \alpha\} \subset A_\alpha := \{Y : \widehat{R}_{K(., Y)}^{\{u > \alpha h\}} = K(., Y)\}.$$

We show that $\mu_h(A_\alpha) = 0$. Consider the measure $\nu = 1_{A_\alpha} \mu_h$ on $\Delta_1(U)$ and the corresponding function

$$v = \int K(., Y) d\nu(Y) = \int \widehat{R}_{K(., Y)}^{\{u > \alpha h\}} d\nu(Y) = \widehat{R}_{K\nu}^{\{u > \alpha h\}} = \widehat{R}_v^{\{u > \alpha h\}},$$

the second equality by (4.1) and the third equality by Lemma 3.21. Since $v \leq \int K(., Y) d\mu_h = h$ and $0 < \alpha < 1$ it follows that

$$v = \widehat{R}_v^{\{u > \alpha h\}} \leq \widehat{R}_h^{\{u > \alpha h\}} \leq \frac{u}{\alpha} \wedge h \leq \frac{u}{\alpha} \wedge \frac{h}{\alpha} = \frac{p}{\alpha}.$$

Because $v = \int K(., Y) d\nu(Y) \in \mathcal{H}_i(U)$ and $p/\alpha \in \mathcal{P}(U)$ we find by Lemma 4.1

(a) (applied to $h_1 = v$, $h_2 = 0$) that $v = 0$, that is,

$$v = \int K(., Y) d\nu(Y) = \int_{A_\alpha} K(., Y) d\mu_h(Y) = 0,$$

and since $K(., Y) > 0$ it follows that $\mu_h(A_\alpha) = 0$. By varying α through a decreasing sequence tending to 0 we conclude from (4.2) that indeed $\mu_h(\{Y : \limsup_{\mathcal{F}(Y)} u/h > 0\}) = 0$. \square

The rest of the proof of Theorem 4.5 proceeds much as in the classical case. For the convenience of the reader we bring most of the details, following in part [4, Section 9.4].

Corollary 4.7. *Let $u, h \in \mathcal{H}_i(U)$, where $h \neq 0$ and μ_u, μ_h are mutually singular. Then $u \wedge h \in \mathcal{P}(U)$, and u/h has minimal-fine limit 0 μ_h -a.e. on $\Delta_1(U)$.*

Proof. There are Borel subsets A_1, A_2 of U such that $A_1 \cup A_2 = \Delta_1(U)$ and $\mu_u(A_1) = \mu_h(A_2) = 0$. Write $u \wedge h = p + k$ with $p \in \mathcal{P}(U)$, $k \in \mathcal{H}_i(U)$. Then $k \leq u$, hence $k \preceq u$ by Lemma 2.2, and so $\mu_k \leq \mu_u$. Similarly, $\mu_k \leq \mu_h$. It follows that $\mu_k(\Delta_1(U)) \leq \mu_u(A_1) + \mu_h(A_2) = 0$ and hence $k = 0$, and so $u \wedge h = p \in \mathcal{P}(U)$. The remaining assertion now follows from Proposition 4.6. \square

Corollary 4.8. *Let $h \in \mathcal{H}_i(U) \setminus \{0\}$, let A be a Borel subset of $\Delta(U)$, and let $h_A = K(1_A \mu_h) = \int_A K(., Y) d\mu_h(Y)$. Then h_A/h has minimal-fine limit $1_A(Y)$ at Y μ_h -a.e. for $Y \in \Delta_1(U)$.*

Proof. Write $u = h - h_A \in \mathcal{H}_i(U)$, which is invariant because $h_A \preceq h$. Since $h_A = K(1_A \mu_h)$ and because $1_A \mu_h$ is carried by $\Delta_1(U)$ along with μ_h , we have $\mu_{h_A} = 1_A \mu_h$, which is carried by $A \cap \Delta_1(U)$. Similarly, $\mu_u = \mu_h - \mu_{h_A} = 1_{U \setminus A} \mu_h$ is carried by $\Delta_1(U) \setminus A$. In particular, μ and μ_h are mutually singular. It follows by Corollary 4.7 that h_A/h and $u/h = 1 - h_A/h$ have minimal fine limit 0 μ_h -a.e. on A and on $\Delta_1(U) \setminus A$, respectively, whence the assertion. \square

Definition 4.9. For any function $h \in \mathcal{H}_i(U) \setminus \{0\}$ and any μ_h -integrable function f on $\Delta_1(U)$ we define

$$u_{f,h}(x) = \int K(x, Y) f(Y) d\mu_h(Y) \quad \text{for } x \in U.$$

Proposition 4.10. *Let $h \in \mathcal{H}_i(U) \setminus \{0\}$ and let f be a μ_h -integrable function on $\Delta(U)$. Then $u_{f,h}/h$ has minimal-fine limit $f(Y)$ at μ_h -a.e. point Y of $\Delta_1(U)$.*

Proof. We may assume that $f \geq 0$. The case where f is a (Borel) step function follows easily from Corollary 4.8. For the general case we refer to the proof of [4, Theorem 9.4.5], which carries over entirely. \square

We are now prepared to prove Theorem 4.5, the Fatou-Naïm-Doob theorem in our setting.

Proof of Theorem 4.5. Write $v = p + h$ with $p \in \mathcal{P}(U)$ and $h \in \mathcal{H}_i(U)$. By our definition of μ_v we then have $\mu_v = \mu_h$. We may assume that $h \neq 0$, for

otherwise $\mu_v = 0$ and the assertion becomes trivial. Let ν be the singular component of μ_u with respect to $\mu_v = \mu_h$. Write $u = q + k$ with $q \in \mathcal{P}(U)$ and $k \in \mathcal{H}_i(U)$. Then $\mu_u = \mu_k = f\mu_h + \nu$, and hence in view of Definition 4.9

$$u = q + u_{f,h} + \int K(., Y) d\nu(Y).$$

By applying Proposition 4.6 with u replaced by q (hence μ_u by $\mu_q = 0$), next Corollary 4.7 with u replaced by $K\nu$ (hence μ_h replaced by ν and $d\mu_u/d\mu_h$ by $d\nu/d\mu_h = 0$), and finally by applying Proposition 4.10 to the present $u_{f,h}$, we see that u/h has minimal-fine limit $f(Y)$ at μ_v -a.e. point Y of $\Delta_1(U)$. Since u/v is defined quasi-everywhere in U and

$$\frac{u}{v} = \frac{u/h}{1 + p/h},$$

the theorem now follows by applying Proposition 4.6 with u there replaced by p (and hence μ_u by the present $\mu_p = 0$). \square

REFERENCES

- [1] Aikawa, H.: *Potential Analysis on non-smooth domains – Martin boundary and boundary Harnack principle*, Complex Analysis and Potential Theory, 235–253, CRM Proc. Lecture Notes 55, Amer. Math. Soc., Providence, RI, 2012.
- [2] Alfsen, E.M.: *Compact Convex Sets and Boundary Integrals*, Ergebnisse der Math., Vol. 57, Springer, Berlin, 2001.
- [3] Ancona, A.: *Sur une conjecture concernant la capacité et l’effilement*, Theorie du Potentiel (Orsay, 1983), Lecture Notes in Math. 1096, Springer, Berlin, 1984, 34–68.
- [4] Armitage, D.H., Gardiner, S.J.: *Classical Potential Theory*, Springer, London, 2001.
- [5] Beznea, L., Boboc, N.: *On the tightness of capacities associated with sub-Markovian resolvents*, Bull. London Math. Soc. **37** (2005), 899–907.
- [6] Boboc, N., Bucur, Gh., Cornea, A.: *Order and Convexity in Potential Theory: H-Cones*, Lecture Notes in Math. 853, Springer, Berlin, 1981.
- [7] Boboc, N., Bucur Gh.: *Natural localization and natural sheaf property in standard H-cones of functions*, I, Rev. Roumaine Math. Pures Appl. **30** (1985), 1–21.
- [8] Constantinescu C. A. Cornea A.: *Potential Theory on Harmonic spaces*, Springer, Heidelberg, 1972.
- [9] Dellacherie, C., Meyer, P.A.: *Probabilités et Potentiel*, Hermann, Paris 1987, Chap. XII–XVI.
- [10] Doob, J.L.: *Applications to analysis of a topological definition of smallness of a set*, Bull. Amer. Math. Soc. **72** (1966), 579–600.
- [11] Doob, J.L.: *Classical Potential Theory and Its Probabilistic Counterpart*, Grundlehren Vol. 262, Springer, New York, 1984.
- [12] El Kadiri, M.: *Sur la décomposition de Riesz et la représentation intégrale des fonctions finement surharmoniques*, Positivity **4** (2000), no. 2, 105–114.
- [13] El Kadiri, M.: *Sur les suites de fonctions finement harmoniques*, Rivista Univ. Parma. **72** (2003), 225–251.
- [14] El Kadiri, M., Fuglede, B.: *Sweeping at the Martin boundary of a finely open set*, Manuscript (2014).

- [15] El Kadiri, M., Fuglede, B.: *The Dirichlet problem at the Martin boundary of a finely open set*, Manuscript (2014).
- [16] Fuglede, B.: *Finely Harmonic Functions*, Lecture Notes in Math. 289, Springer, Berlin, 1972.
- [17] Fuglede, B.: *Remarks on fine continuity and the base operation in potential theory*, Math. Ann. **210** (1974), 207–212.
- [18] Fuglede, B.: *Sur la fonction de Green pour un domaine fin*, Ann. Inst. Fourier **25**, 3–4 (1975), 201–206.
- [19] Fuglede, B.: *Finely harmonic mappings and finely holomorphic functions*, Ann. Acad. Sci. Fennicae, Ser. A.I. **10** (1976), 113–127.
- [20] Fuglede, B.: *Localization in fine potential theory and uniform approximation by subharmonic functions*, J. Funct. Anal. **49** (1982) 52–72.
- [21] Fuglede, B.: *Integral representation of fine potentials*, Math. Ann. **262** (1983), 191–214.
- [22] Fuglede, B.: *Représentation intégrale des potentiels fins*, C.R. Acad. Sc. Paris **300**, Ser. I, 5 (1985), 129–132.
- [23] Gardiner, S.J., Hansen, W.: *The Riesz decomposition of finely superharmonic functions*, Adv. Math. **214**, 1 (2007), 417–436.
- [24] Gowrisankaran, K.: *Extreme harmonic functions and boundary value problems*, Ann. Inst. Fourier **13**, 2 (1963), 307–356.
- [25] Hervé, R.-M.: *Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier **12** (1962), 415–571.
- [26] Le Jan, Y.: *Quasi-continuous functions associated with Hunt processes*, Proc. Amer. Math. Soc. **86** (1982), 133–138.
- [27] Le Jan, Y.: *Quasi-continuous functions and Hunt processes*, J. Math. Soc. Japan **35** (1983), 37–42.
- [28] Le Jan, Y.: *Fonctions cad-lag sur les trajectoires d'un processus de Ray*, Theorie du Potentiel (Orsay, 1983), Lecture Notes in Math. 1096 (Springer, 1984), 412–418.
- [29] Lyons, T.: *Cones of lower semicontinuous functions and a characterisation of finely hyperharmonic functions*, Math. Ann. **261** (1982), 293–297.
- [30] Mokobodzki, G.: *Représentation intégrale des fonctions surharmoniques au moyen des réduites*, Ann. Inst. Fourier **15**, 1 (1965), 103–112.
- [31] Taylor, J.C.: *An elementary proof of the theorem of Fatou-Naïm-Doob*, Canadian Mathematical Society Conference Proceedings, Vol. 1, 1981.

UNIVERSITÉ MOHAMMED V, DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES, B.P. 1014, RABAT, MOROCCO
E-mail address: elkadiri@fsr.ac.ma

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN, DENMARK
E-mail address: fuglede@math.ku.dk